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A lower bound for the modulus of the Dirichlet eta function on partition $\mathcal{P}$ from 2-D principal component analysis

Yuri Heymann

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Abstract The present manuscript aims to derive an expression for the lower bound of the modulus of the Dirichlet eta function on vertical lines $\Re(s) = \alpha$. An approach based on a two-dimensional principal component analysis matching the dimensionality of the complex plane, which is built on a parametric ellipsoidal shape, has been undertaken to achieve this result. This lower bound, which is expressed as $\forall s \in \mathbb{C}$ s.t. $\Re(s) \in \mathcal{P}(s)$, $|\eta(s)| \geqslant |1 - \frac{\sqrt{2}}{2^\alpha}|$, where η is the Dirichlet eta function, has implications for the Riemann hypothesis as $|\eta(s)| > 0$ for any such $s \in \mathcal{P}$, where $\mathcal{P}$ is a partition spanning one half of the critical strip, on either sides of the critical line $\Re(s) = 1/2$ depending upon a variable delimiting regions, which are complementary by mirror symmetry with respect to $\Re(s) = 1/2$.

Keywords Dirichlet eta function, PCA, Analytic continuation

1 Introduction

The Dirichlet eta function is an alternating series related to the Riemann zeta function of interest in the field of number theory for the study of the distribution of primes [12]. Both series are tied together on a two-by-two relationship expressed as $\eta(s) = \left(1 - 2^{1-s}\right) \zeta(s)$ where s is a complex number. The location of the non-trivial zeros of the Riemann zeta function in the critical strip $\Re(s) \in]0, 1[$ is key in the prime-number theory. For example, the Riemann-von Mangoldt explicit formula, as an asymptotic expansion of the prime-counting function, involves a sum over the non-trivial zeros of the Riemann zeta function [10]. The Riemann hypothesis, which scope is the domain of existence of the zeros in the critical strip, has implications for the accurate estimate of the error involved in the prime-number theorem and a variety of conjectures such as the

Yuri Heymann
E-mail: y.heymann@yahoo.com
Address in der Schweiz: 3 rue Chandieu, 1202 Geneva, Switzerland

Lindelöf hypothesis [2], conjectures about short intervals containing primes [9], Montgomery's pair correlation conjecture [8], the inverse spectral problem for fractal strings [5], etc. Moreover, variants of the Riemann hypothesis falling under the generalized Riemann hypothesis in the study of modular L-functions [11] are core for many fundamental results in number theory and related fields such as the theory of computational complexity. For instance, the asymptotic behavior of the number of primes less than x described in the prime-number theorem, $\pi(x) \sim \frac{x}{\ln(x)}$, provides a smooth transition of time complexity as x approaches infinity. As such, the time complexity of the prime-counting function using the $x/(\ln x)$ approximation is of order $\mathcal{O}(M(n) \log n)$, where n is the number of digits of x and $M(n)$ is the time complexity for multiplying two n-digit numbers. This figure is based on the time complexity to compute the natural logarithm with the arithmetic-geometric mean approach where n represents the number of digits of precision.

The below definitions are provided on a informal basis as a supplement to standard definitions when referring to reals, complex numbers and holomorphic functions. A complex number is the composite of a real and imaginary number, forming a 2-D surface, where the pure imaginary axis is represented by the letter i such that $i^2 = -1$. The neutral element and index i form a basis spanning some sort of vector space. The Dirichlet eta function is a holomorphic function having for domain a subset of the complex plane where reals are positive denoted $\mathbb{C}^+$, which arguments are sent to codomain in $\mathbb{C}$. As an holomorphic function it is characterized by its modulus, a variable having for support an axial vertex orthonormal to the complex plane. The conformal way to describe space in geometrical terms is the orthogonal system, consisting of eight windows delimited by the axes of the Cartesian coordinates, resulting from the union of the three even surfaces of Euclidean space.

A Hilbert space as an extension of the former, is a multidimensional space which in current context of some functions in $\mathcal{L}^2$ space referring to squared-integrable functions, is comprised of wave functions expressing components or basis elements of space, further equipped of an inner product defining a norm and angles between these functions. The well-known Riemann hypothesis as the eighth problem David Hilbert presented at the International Congress of Mathematicians in Paris in 1900 [4], has strong relation with such Hilbertian spaces, as unveiled in the remaining of the manuscript.

As a reminder, the definition of the Riemann zeta function and its analytic continuation to the critical strip are displayed below. The Riemann zeta function is commonly expressed as follows:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} , \tag{1}$$

where s is a complex number and $\Re(s) > 1$ by convergence in the above expression.

The standard approach for the analytic continuation of the Riemann zeta function to the critical strip $\Re(s) \in]0,1[$ is performed with the multiplication of $\zeta(s)$ with the function $\left(1 - \frac{2}{2^s}\right)$, leading to the Dirichlet eta function. By definition, we have:

$$\eta(s) = \left(1 - \frac{2}{2^s}\right)\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}\,, \tag{2}$$

where $\Re(s) > 0$ and η is the Dirichlet eta function. By continuity as s approaches one, $\eta(1) = \ln(2)$.

The function $\left(1 - \frac{2}{2^s}\right)$ has an infinity of zeros on the line $\Re(s) = 1$ given by $s_k = 1 + \frac{2k\pi i}{\ln 2}$ where $k \in \mathbb{Z}^*$. As $\left(1 - \frac{2}{2^s}\right) = 2 \times \left(2^{\alpha-1}\, e^{i\,\beta \ln 2} - 1\right) / \, 2^{\alpha}\, e^{i\,\beta \ln 2}$, the factor $\left(1 - \frac{2}{2^s}\right)$ has no poles nor zeros in the critical strip $\Re(s) \in]0,1[$. As such, the Dirichlet eta function can be used as a proxy of the Riemann zeta function for zero finding in the critical strip $\Re(s) \in]0,1[$.

From the above, the Dirichlet eta function is expressed as:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, e^{-i\,\beta\ln(n)}}{n^{\alpha}}\,, \tag{3}$$

where $s = \alpha + i\,\beta$ is a complex number, α and β are real numbers.

We have $\frac{1}{n^s} = \frac{1}{n^{\alpha}\exp(\beta i \ln n)} = \frac{1}{n^{\alpha}(\cos(\beta \ln n) + i \sin(\beta \ln n))}$. We then multiply both the numerator and denominator by $\cos(\beta\ln(n)) - i\sin(\beta\ln(n))$. After several simplifications, $\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}[\cos(\beta \ln n) - i \sin(\beta \ln n)]}{n^{\alpha}}$.

In the remaining of the manuscript, the Riemann zeta function is referring to its formal definition and analytic continuation by congruence.

2 Mathematicals

2.1 Elementary propositions no. 1 $\sim$ 5

Proposition 1 Given z_1 and z_2 two complex numbers, we have:

$$\Big| z_1 + z_2 \Big| \geqslant \Big| |z_1| - |z_2| \Big|\,, \tag{4}$$

where $|z|$ denotes the modulus of the complex number z. As a variant of the reverse triangle inequality, (4) can be extended to any normed vector space, where the norm is subadditive over its domain of definition, see [6,7].

Proof As a demonstration of the reverse triangle inequality in complex domain equipped of a norm using Euler's notations, we introduce z_1 and z_2 in polar coordinates. We have $z_1 = r_1 e^{i\,\theta_1}$ and $z_2 = r_2 e^{i\,\theta_2}$, where their respective modulus and the constituents of imaginary arguments, θ_1 and θ_2, are reals. We get:

$$\begin{aligned}\left|z_1+z_2\right| &= \left|r_1 e^{i\,\theta_1}+r_2 e^{i\,\theta_2}\right| \\ &= \sqrt{\left(r_1\cos(\theta_1)+r_2\cos(\theta_2)\right)^2+\left(r_1\sin(\theta_1)+r_2\sin(\theta_2)\right)^2} \\ &= \sqrt{r_1^2+r_2^2+2r_1r_2\cos(\theta_1+\theta_2)}\,.\end{aligned} \tag{5}$$

Trigonometric identities $2\cos(a)\cos(b) = \cos(a-b)+\cos(a+b)$ and $2\sin(a)\sin(b) = \cos(a-b)-\cos(a+b)$ are invoked as part of (5), see [1] formulas 4.3.31 and 4.3.32, p. 72. Reverse triangle inequality (4) follows from (5) and (6), as $\forall\theta, 1+\cos(\theta) \geqslant 0$.

$$\begin{aligned}\Big||z_1|-|z_2|\Big| &= \left|r_1+r_2 e^{i\,\pi}\right| \\ &= \sqrt{r_1^2+r_2^2-2r_12_2}\,.\end{aligned} \tag{6}$$

Proposition 2 Let us consider an ellipse $(x_t, y_t) = [a\cos(t), b\sin(t)]$ where a and b are two positive reals corresponding to the lengths of the semi-major and semi-minor axes of the ellipse $(a \geqslant b)$ and $t \in [0, 2\pi]$ is a variable having a correspondance with the angle between the x-axis and the vector (x_t, y_t).

Let us set t such that the semi-major axis of the ellipse is aligned with the x-axis, which is the angle maximizing the objective function defined as the modulus of (x_t, y_t). When $|(x_t, y_t)|$ is maximized, we have:

$$|(x_t, y_t)| = x_t + y_t = a\,. \tag{7}$$

Note that by maximizing $x_t + y_t$, we would get $|(x_t, y_t)| < x_t + y_t$, as the expression x_t+y_t is maximized when $t = \arctan(b/a)$, leading to $\max(x_t+y_t) = \sqrt{a^2+b^2}$.

Proof Principal Component Analysis (PCA), is a statistical method for reducing the dimensionality of a variable space by representing it with a few orthogonal variables capturing most of the variability of an observable. In the current context, a two-dimensional principal component analysis built on a parametric ellipsoidal shape is introduced to match the dimensionality of the complex plane. An ellipse centered at the origin of the coordinate system can be parametrized as follows: $(x_t, y_t) = [a\cos(t), b\sin(t)]$ where a and b are positive real numbers corresponding to the lengths of the semi-major and semi-minor axes of the ellipse $(a \geqslant b)$ and $t \in \mathbb{R}$ is a variable having correspondance with the angle between the x-axis and the vector (x_t, y_t). As the directions of the x and y-axes are orthogonal, the objective function is maximized with respect to t when the major axis is aligned with the x-axis. When $|(x_t, y_t)|$ is at its maximum value, we get $|(x_t, y_t)| = x_t + y_t = a$ where a is the length of the semi-major axis. The modulus of (x_t, y_t) is as follows:

$|(x_t, y_t)| = \sqrt{a^2 \cos^2(t) + b^2 \sin^2(t)} \leqslant a, \forall t \in \mathbb{R}$. By analogy with principal component analysis, x_t represents the first principal component and y_t the second principal component. Say x_t and y_t were not orthogonal, then there would be a non-zero phase shift φ between the components, i.e. $x_t = a\cos(t)$ and $y_t = b\sin(t + \varphi)$.

Proposition 3 Given a vector $V_{\mathcal{E}} = [u(\mathcal{E}), v(\mathcal{E})]$ defined in a bidimensional vector space, where $u(\mathcal{E})$ and $v(\mathcal{E})$ are two real-valued functions say on $\mathbb{R}^\nu \to \mathbb{R}$, where ν represents the degrees of freedom of the system. The reference of a point in such system, is described by the set $\mathcal{E} = \{\varepsilon_1, \varepsilon_2, ..., \varepsilon_\nu\}$ representing a multidimensional coordinate sytem. Thus, we have:

$$|V_{\mathcal{E}}| = u(\mathcal{E}) + v(\mathcal{E}) \,, \tag{8}$$

if only $u(\mathcal{E})\, v(\mathcal{E}) = 0$ and $u(\mathcal{E}) + v(\mathcal{E}) \geqslant 0$.

Given a basis set $\{e_1, e_2\}$ of the above-mentioned vector space, where $|e_i| = 1$ for $i = 1, 2$, *proposition 3* is true if only the inner product across basis elements is equal to zero, i.e. $e_1 \cdot e_2 = 0$, meaning that the basis elements are disentangled from each other. We say that $\{e_1, e_2\}$ is an orthonormal basis. This condition is also necessary for *propositions 2 and 4* to be true, in 2-D Cartesian frame.

Proof By the square rule, we have $(u + v)^2 = u^2 + v^2 + 2\, u\, v$. The modulus of a vector V as defined in such two-dimensional frame is $|V| = \sqrt{u^2 + v^2}$, leading to $|V| = |u + v|$ if only $u\, v = 0$, which is provided as a complement to *proposition 2*. The above as a support of pre-Hilbertian spaces by the scalar product $u\, v$, is a prerequisite for Hilbertian spaces of squared-integrable functions referring to such $\mathcal{L}^2$ spaces equipped of an inner product.

Proposition 4 Given a circle of radius $r \in \mathbb{R}^+$ parametrized as follows: $(x_t, y_t) = [r\, \cos(t), r\, \sin(t)]$ where t is a real variable in $[0, 2\pi]$, we construct a function $f(t) = a\cos(t) + b\sin(t + \varphi)$ where a and b are two positive reals and φ a real variable which can be positive or negative such that:

$$r\, \cos(t) + r\, \sin(t) = a\cos(t) + b\sin(t + \varphi) \,, \tag{9}$$

$\forall t \in \mathbb{R}$ and where φ is a real variable of t (when a and b are scalars).

As an excerpt of below proof elements, $\forall t \in [0, 2\pi]$ and $\forall \delta \in [-r, r]$ we have:

$r\cos(t) + r\sin(t) = (r + \delta)\cos(t) + \sqrt{r^2 + \delta^2}\sin(t + \varphi)$, where $\varphi = -\arctan \delta/r$, making φ independent of t by some correspondance between a, b and φ represented as parametric functions of δ and where $\delta/r \in [-1, 1]$.

As such $a = r + \delta$ and $b = \sqrt{r^2 + \delta^2}$ where $\delta = -r\, \tan(\varphi)$, yielding $a/r = 1 - \tan\varphi$ and $b/r = \sqrt{1 + \tan^2\varphi}$ where $\varphi \in]-\pi/4, \pi/4[$.

Say $u_t = a\cos(t)$ and $v_t = b\sin(t+\varphi)$.

When $a \geqslant b$, the first component u_t is the one carrying most of the variance of expression $f(t)$, meaning it is leading component. Thus, we have:

$$|(x_t, y_t)| \leqslant \max(v_t) \leqslant \max(u_t)\,, \tag{10}$$

$\forall t \in [0, 2\pi]$, where $\max(u_t)$ is the maximum value of u_t and $\max(v_t)$ the maximum value of v_t over the interval $[0, 2\pi]$.

When $a \leqslant b$, the component v_t carries most of the variance of $f(t)$ meaning it is leading component, and we have:

$$\max(u_t) \leqslant |(x_t, y_t)| \leqslant \max(v_t)\,, \tag{11}$$

$\forall t \in [0, 2\pi]$, where $\max(u_t)$ and $\max(v_t)$ as above.

When the functions u_t and v_t are orthogonal i.e. $\varphi = 0$, we have $r = a = b$.

Proof Given that $(r+\delta)\cos(t) + (r-\delta_2)\sin(t) = r\cos(t) + r\sin(t) + \delta\cos(t) - \delta_2\sin(t)$, where δ and δ_2 are variables in $\mathbb{R}$ i.e. sensitive to t. As we want $\delta\cos(t) - \delta_2\sin(t) = 0$, we have $\delta_2 = \delta\cot(t)$. Thus, we get: $r\cos(t) + r\sin(t) = (r+\delta)\cos(t) + (r - \delta\cot(t))\sin(t)$. As $r\sin(t) - \delta\cos(t) = \sqrt{r^2+\delta^2}\sin(t+\varphi)$ where $\varphi = -\arctan\delta/r$, we get $\forall t \in [0, 2\pi]$, $r\cos(t) + r\sin(t) = (r+\delta)\cos(t) + \sqrt{r^2+\delta^2}\sin(t+\varphi)$. We set $a = r+\delta$ and $b = \sqrt{r^2+\delta^2}$ where $-r \leqslant \delta \leqslant r$, leading to (9). If $\delta \geqslant 0$, we have $r \leqslant \sqrt{r^2+\delta^2} \leqslant r+\delta$, leading to (10). If $\delta \leqslant 0$, we have $r+\delta \leqslant r \leqslant \sqrt{r^2+\delta^2}$, leading to (11). As $\max\{r\cos(t) + r\sin(t)\} = \sqrt{2}\,r$ which occurs when $t = \frac{\pi}{4}$, we have $\delta \in [-r, r]$ for any $r \geqslant 0$.

Proposition 5 Given an alternating series S_m constructed on a sequence $\{a_n\}$ monotonically decreasing with respect to its index $n \geqslant m \in \mathbb{N}$ where $a_n > 0$ and $\lim_{n\to\infty} a_n = 0$, defined such that:

$$S_m = \sum_{n=m}^{\infty} (-1)^{n+1}\, a_n\,, \tag{12}$$

where m is an integer, we get the below upper bound inequality on the absolute value of the series:

$$\left|\sum_{n=m}^{\infty} (-1)^{n+1}\, a_n\right| \leqslant a_m\,, \tag{13}$$

where a_m represents a radius in Leibniz's notations.

Proof Given Leibniz's rule, the series S_m is convergent as $\{a_n\}$ is monotonically decreasing and $\lim_{n\to\infty} a_n = 0$. Let us define the series $L = \sum_{n=1}^{\infty}(-1)^{n+1}a_n$ and its partial sum $S^k = \sum_{n=1}^{k}(-1)^{n+1}a_n$. The odd partial sums decrease as $S^{2(m+1)+1} = S^{2m+1} - a_{2m+2} + a_{2m+3} \leqslant S^{2m+1}$. The even partial sums increase as $S^{2(m+1)} = S^{2m} + a_{2m+1} - a_{2m+2} \geqslant S^{2m}$. As the odd and even partial sums converge to the same value, we have $S^{2m} \leqslant L \leqslant S^{2m+1}$ for any finite $m \in \mathbb{N}^*$.

When m is odd:

$$S_m = a_m + \sum_{n=m+1}^{\infty} (-1)^n \, a_n = a_m + (L - S^m) \leqslant a_m \, . \tag{14}$$

When m is even:

$$S_m = -a_m + \sum_{n=m+1}^{\infty} (-1)^n \, a_n = -a_m + (L - S^m) \geqslant -a_m \, . \tag{15}$$

Leading to:

$$|S_m| \leqslant a_m \, , \tag{16}$$

where m is a natural number, finite in $\mathbb{N}$*.

2.2 The lower bound of the Dirichlet eta modulus as a floor function

In standard notations, the point s is expressed as $s = \alpha + i\,\beta$ where α and β are reals in their corresponding basis belonging to $\mathbb{C}$.

Note the zeros of the Dirichlet eta function and its complex conjugate are the same. For convenience, we introduce the conjugate of the Dirichlet eta function, expressed as follows:

$$\overline{\eta}(s) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{e^{i\,\beta \ln n}}{n^{\alpha}} \, , \tag{17}$$

where $\Re(s) > 0$. By applying *proposition 1* to (17), we get:

$$|\overline{\eta}(s)| \geqslant \left| 1 - \left[\sum_{n=2}^{\infty} (-1)^{n+1} \frac{e^{i\,\beta \ln n}}{n^{\alpha}} \right] \right| \, , \tag{18}$$

where $[z]$ denotes the norm of the complex number z. The square brackets are used as smart delimiters for operator precedence.

With respect to the expression $\left| \sum_{n=2}^{\infty} (-1)^{n+1} \frac{e^{i\,\beta \ln n}}{n^{\alpha}} \right|$, its decomposition into sub-components $u_n = \frac{(-1)^{n+1}}{n^{\alpha}} e^{i\beta \ln n}$ is a vector representation where $\beta \ln n + (n+1)\pi$ is the angle between the real axis and the orientation of the vector itself, and where $\frac{1}{n^{\alpha}}$ is its modulus. The idea is to apply a rotation by an angle θ to all component vectors simultaneously, resulting in a rotation of the vector of their sum. The resulting vector after rotation θ expressed in Euler's notation is $v_{\theta,\beta} = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^{\alpha}} e^{i(\,\beta \ln n + \theta)}$, where θ and β are real numbers.

Let us introduce the objective function w, defined as the sum of the real and imaginary parts of $v_{\theta,\beta}$, i.e. $w = v_x + v_y$ where $v_x = \Re(v_{\theta,\beta})$ and $v_y = \Im(v_{\theta,\beta})$. We get:

$$\begin{aligned} w &= \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^{\alpha}} \left(\cos(\beta \ln n + \theta) + \sin(\beta \ln n + \theta)\right) \\ &= \sum_{n=2}^{\infty} (-1)^{n+1} \frac{\sqrt{2}}{n^{\alpha}} \cos\left(\beta \ln n + \theta - \frac{\pi}{4}\right) . \end{aligned} \tag{19}$$

The trigonometric identity $\cos(x)+\sin(x) = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$ which follows from $\cos(a)\cos(b) + \sin(a)\sin(b) = \cos(a - b)$ with $b = \frac{\pi}{4}$ is invoked in (19), see [1] formulas 4.3.31 and 4.3.32, p. 72. The finite sum of a subset of the elements of the second line of (19) from 2 to $n \in \mathbb{N}$ is further referred to as the w-series.

In the remainder of the manuscript, orthogonality between functions is defined in terms of the inner product. While orthogonality between vectors is defined in terms of the scalar product between such pairs, for real functions on $\mathbb{R}$ to $\mathbb{R}$ we usually define an integration product forming an $\mathcal{L}^2$ space. Let us say we have two real-valued functions f and g, which are squared-*Lebesgue* integrable on a segment $[a, b]$ and where the inner product between f and g is given by:

$$\langle f, g \rangle = \int_a^b f(x)\, g(x) dx\,. \tag{20}$$

The functions f and g are squared-*Lebesgue* integrable, meaning such functions can be normalized i.e. the squared norm as defined by $\langle f, f \rangle$ is finite. For sinusoidal functions such as sine and cosine, it is common to say $[a, b] = [0, 2\,\pi]$, which interval corresponds to one period. The condition for functions f and g to be orthogonal is that the inner product as defined in (20) is equal to zero.

The objective function w was constructed by adding together the real and imaginary parts of $v_{\theta,\beta}$. We note that the real and imaginary parts of $v_{\theta,\beta}$ are orthogonal due to Euler's formula. Hence, $\Re(v_{\theta,\beta})$ is maximized in absolute value when $\Im(v_{\theta,\beta}) = 0$, which occurs for example when $\beta = 0$. As $w_{\theta=\frac{\pi}{4},\beta} = \sqrt{2}\,\Re(v_{\theta=0,\beta})$, the maximum value of the objective function w provided that $\theta = \frac{\pi}{4}$ occurs when $\beta = 0$, leading to:

$$max_{\star}\{|w|\} = \left| \sum_{n=2}^{\infty} (-1)^{n+1} \frac{\sqrt{2}}{n^{\alpha}} \right| , \tag{21}$$

which is also the maximum value of the objective function w, given that $\beta = 0$. Although there could be other values of the tuple (θ, β) leading to larger values of $max\{|w|\}$, a particularity of expression $max_{\star}\{|w|\}$ as defined above is to be equal to $\sqrt{2}\,\left|v_{\theta=\frac{\pi}{4},\beta=0}\right|$.

By applying *proposition 5* to (21), we get $max_\star\{|w|\} \leqslant \frac{\sqrt{2}}{2^\alpha}$, leading to the below inequality:

$$\left|v_{\theta=\frac{\pi}{4},\beta=0}\right| \leqslant \frac{1}{2^\alpha} < \frac{\sqrt{2}}{2^\alpha}\,, \tag{22}$$

which is less than the maximum of $|v_{\theta,\beta}|$, while the inner components of $v_{\theta,\beta}$ collinear as $\beta = 0$.

We proceed with the decomposition of the objective function w into dual components w_1 and w_2, expressed as follows:

$$w_1 = -\frac{\sqrt{2}}{2^\alpha}\cos\left(\beta\ln(2) + \theta - \frac{\pi}{4}\right)\,, \tag{23}$$

and

$$w_2 = \sum_{n=3}^{\infty}(-1)^{n+1}\frac{\sqrt{2}}{n^\alpha}\cos\left(\beta\ln n + \theta - \frac{\pi}{4}\right)\,, \tag{24}$$

where most of the variance of the w-series comes from the leading component.

By construction w is the sum of the real and imaginary parts of $v_{\theta,\beta}$ which are orthogonal functions. Let us say v_θ is the parametric notation of $v_{\theta,\beta}$ for a given β value. We note that for any given β value, the complex number v_θ describes a circle in the complex plane, centered on the origin. Hence, in symbolic notations, w can be written as $w = r\cos(t)+r\sin(t)$. The components w_1 and w_2 can be expressed as $w_1 = a\cos(t)$ and $w_2 = b\sin(t+\varphi)$ where t is a variable in $[0, 2\pi]$ and φ some variable in $\mathbb{R}$. As we suppose that w_1 carries most of the variance of w (i.e. $a \geqslant b$), the modulus $|v_\theta|$ is smaller or equal to the maximum value of $|w_1|$, by *proposition 4*. Yet, $|v_\theta|$ is equal to $\max\{w_1\}$, if w_1 and w_2 are orthogonal and $t = 0$. By *proposition 2*, at its maximum value $|(w_1, w_2)| = w_1 + w_2$ when $\varphi = 0$ and $t = 0$, which in light of the above, is also equal to the maximum value of $|v_\theta|$. As the inner product between w_1 and w_2 does not depend on θ, orthogonality between w_1 and w_2 is determined by β values. We then apply a rotation by an angle θ to maximize the objective function $|(w_1, w_2)|$. We consider two complementary scenarios respectively, depending whether w_1 is the leading component of the w-series or some other function i.e. w_2 as the alternative by dual decomposition of w.

When w_1 is leading component:

Say w_1 is leading component in some regions of the critical strip denoted $\mathcal{A}$. In this scenario, as we suppose w_1 and w_2 are orthogonal at the maximum value of $|(w_1, w_2)|$, i.e. $w_1 = \frac{\sqrt{2}}{2^\alpha}$ and $w_2 = 0$, we get $max\{|(w_1, w_2)|\} = \frac{\sqrt{2}}{2^\alpha}$. If we suppose that w_1 and w_2 are not orthogonal, by *proposition 4* we would get $\left|\sum_{n=2}^{\infty}(-1)^{n+1}\frac{e^{i\,\beta\ln n}}{n^\alpha}\right| < \frac{\sqrt{2}}{2^\alpha}$, and $|\eta(s)|$ would be strictly larger than zero when $\alpha = 1/2$ in (26). This would imply that the Dirichlet eta function does not have zeros on the critical line $\Re(s) = 1/2$, which is known to be false.

Hence, we can say that when w_1 is the leading component, the functions w_1 and w_2 are orthogonal at the maximum value of $|v_\theta|$, i.e. w_1 is first principal component. Thus, we have:

$$\forall s \in \mathbb{C}^+ \text{ s.t. } s \in \mathcal{A}, \left|\sum_{n=2}^{\infty}(-1)^{n+1}\frac{e^{i\,\beta\ln n}}{n^\alpha}\right| \leqslant \frac{\sqrt{2}}{2^\alpha}, \tag{25}$$

where $\mathbb{C}^+$ is the subset of $\mathbb{C}$ s.t. $\Re(s) > 0$ (acronym s.t. standing for "such that").

Say for $\Re(s) = \alpha \geqslant \frac{1}{2}$, (18) and (25) imply that :

$$|\eta(s)| \geqslant 1 - \frac{\sqrt{2}}{2^\alpha}, \tag{26}$$

for any $s \in \mathcal{A}$ s.t. $\Re(s) = \alpha \in [1/2, \infty[$, where $\mathcal{A} \subseteq \mathbb{C}^+$ i.e. w_1 is leading.

When w_2 is leading component:

As a complementary of the former, say $\mathcal{A}^c$ is some regions of the critical strip where w_1 is not leading component. In this scenario, we fall on eq. (11) of *proposition 4*, yielding:

$$\forall s \in \mathbb{C}^+ \text{ s.t. } s \in \mathcal{A}^c, \left|\sum_{n=2}^{\infty}(-1)^{n+1}\frac{e^{i\,\beta\ln n}}{n^\alpha}\right| \geqslant \frac{\sqrt{2}}{2^\alpha}, \tag{27}$$

where $\mathbb{C}^+$ is defined as above.
Say for positive $\Re(s) = \alpha \leqslant \frac{1}{2}$, (18) and (27) imply that:

$$|\eta(s)| \geqslant \frac{\sqrt{2}}{2^\alpha} - 1, \tag{28}$$

for any $s \in \mathcal{A}^c$ s.t. $\Re(s) = \alpha \in]0, 1/2]$, where $\mathcal{A}^c \subseteq \mathbb{C}^+$ seen above.

2.3 Further considerations

The below are based on the premise that the Dirichlet eta function can be used as a proxy of the Riemann zeta function for zero finding in the critical strip, and interpretations about the lower bound of the modulus of the Dirichlet eta function as a floor function. The surface spanned by the modulus of the Dirichlet eta function is a continuum resulting from the application of a real-valued function over the dimensions of the complex plane, which is a planar representation where the reals form a line continuous to the right i.e. an ad-dextram vertice in some latin scriptures, contiguous with the imaginary axis, and where the square of imaginary numbers are subtracted from zero. The modulus of the Dirichlet eta function is a holographic function sending a complex number into a real number, whereas the floor function is a projection of the former onto the real axis.

In the common scenario when w_1 is the leading component, the floor function of the modulus of the Dirichlet eta function on vertical lines $\Re(s) = \alpha$ does not depend on β, which is reflected by the linear relationship between θ and β in the cosine argument of the first principal component, as a single term of w-series. This is no longer the case, when adding together several terms of the w-series as first principal component. As a complementary of the former, scenarios when w_1 is not leading component i.e. w_2 is as the alternative, exist in various parts of the domain as reported by Vincent Grandville (such a point is given by $s = 0.75 + 580.13\,i$ for example). For such scenarios, though there is no straight-forward linear relationship between θ and β of the arguments of component w_2 as the alternative, for $L_\alpha = \left|1 - \sqrt{2}/2^\alpha\right|$ to qualify as floor function in all regions of the domain, L_α is floor function of the modulus of the Dirichlet by mirror symmetry with respect to line $\Re(s) = 1/2$, c.f. (28) in complement of (26). By perfect matching principle under mirror symmetry, components w_1 and w_2 at zeros of the Dirichlet eta function resulting from the continuity to the right of the critical line as given by (27), are meant to match corresponding components at such zeros as given by (25) at the limit to the left of line $\Re(s) = 1/2$. As components w_1 and w_2 are inverted when approaching the critical line from both sides, meaning continuity is violated, this suggests there is no such regions where w_1 is not leading component which is contiguous with the critical line.

A special case of polynomial made up of a subset of the cosine terms of the w-series which does not depend on β occurs, if there exists such a polynomial which is equal to zero for any β. For such a polynomial to be first principal component involves subsequent components are also equal to zero, leading to the degenerate case $|v_\theta| = 0$. This occurs when α tends to infinity, leading to $w = 0$ for β real, a special case of the Dirichlet eta function converging towards unity.

The combination of multiple terms of the w-series as principal component involves that such component is a function composed of terms of the form $a_n = \pm\frac{1}{n^\alpha}\cos(\beta\ln(n) + \theta)$, where n is the index of the corresponding term in v_θ. Due to the multiplicity of bivariate collinear arguments in the cosine functions, which comovements are not parallel across the index n (as a finite set), there is no straight-forward bijection between θ and β, i.e. a one-degree of freedom relationship, such that all cosine arguments $\beta\ln n + \theta$ of the component are decoupled from β. As aforementioned, the lower bound of the modulus of the Dirichlet eta function needs to be decoupled from β, to be a floor function on vertical lines. Moreover, the principal components involved in modulus maximisation need to be disentangled for PCA to be applicable, which in current context means the maximum of $|(w_1, w_2)|$ is reached at orthogonality between w_1 and w_2, a requirement for L_α to be a floor function of the modulus of the Dirichlet eta function. As a rule of thumb, one degree of freedom is needed for every additional principal component, when matching the dimensionality of the variable space in the parametric ellipsoidal model.

By the Riemann zeta functional, if $|\eta(s)| > 0$ on either sides of the critical strip, then the same applies to the other side i.e. $|\eta(1-\bar{s})| > 0$ by mirror symmetry with respect to $\Re(s) = 1/2$. This stems from *proposition 7* as per excerpt in [3], i.e. Given s a complex number and $\bar{s}$ its conjugate, if s is a zero of the Riemann zeta function in the strip $\Re(s) \in]0, 1[$, then $1-\bar{s}$ is also a zero of the function. By complementary with the former, if s is not a zero of the Riemann zeta function in the critical strip, then $1-\bar{s}$ is not a zero as well.

When $\alpha = 1/2$, $|\eta(s)| \geqslant 0$ means the Dirichlet eta function can have some zeros on the critical line $\Re(s) = 1/2$, which is known to be true [13], p. 256. For any $\alpha \in]0, \frac{1}{2}[\, U \,]\frac{1}{2}, 1[, |\eta(s)| > 0$ means the Dirichlet eta function has no zeros on such vertical lines as given by $\Re(s) = \alpha$. As the Dirichlet eta function and the Riemann zeta function share the same zeros in the critical strip, involves the Riemann zeta function has no zeros on such lines $\Re(s) = \alpha$. As the critical strip $\Re(s) \in]0, 1[$ is partitioned into complementary regions depending whether w_1 is leading component or not, (26) and (28) imply that $\forall s \in \mathbb{C}$ s.t. $\Re(s) \in]1/2, 1[$ either $|\eta(s)| \geqslant |1 - \frac{\sqrt{2}}{2^\alpha}|$ or $|\eta(1-\bar{s})| \geqslant |1 - \frac{\sqrt{2}}{2^{1-\alpha}}|$ is true, or both.

Say for any point s in the critical strip, a binary variable determines whether w_1 is leading component or w_2 as the alternative. For α real part spanning $\Re(s) \in]\frac{1}{2}, 1[$, we have $|\eta(s)| \geqslant |1 - \frac{\sqrt{2}}{2^\alpha}|$ or $|\eta(1-\bar{s})| \geqslant |1 - \frac{\sqrt{2}}{2^{1-\alpha}}|$ as the alternative, depending upon this variable. This is similar to saying that $\forall s \in \mathbb{C}$ s.t. $\Re(s) \in \mathcal{P}(s)$, $|\eta(s)| \geqslant |1 - \frac{\sqrt{2}}{2^\alpha}|$, where $\mathcal{P}$ is a partition spanning one half of the critical strip on either sides of the critical line depending upon such a variable delimiting regions, which are complementary by mirror symmetry with respect to $\Re(s) = 1/2$. In 2-d world, this variable is expressed as a function of coordinates in $\mathbb{C}$, which by the frontier with its dual as the complementary of the former, defines the delimitations of the regions where w_1 is not leading component. The above and *proposition 7* in [3], imply the Dirichlet eta function does not have zeros on either parts of the critical strip, i.e. when $\Re(s) \in]0, 1/2[$ and $]1/2, 1[$, which is a prerequisite to say the non-trivial zeros of the Riemann zeta function lie on the critical line $\Re(s) = 1/2$.

A further observation is that in scenarios when w_1 is the leading component, the second term of the Dirichlet eta function is orthogonal to the vector comprised of the remaining terms of the series of index larger than 2 at zeros of the function, by the Hadamar product. Furthermore, the angle between the axis defined by the second component of the Dirichlet eta function and some vector comprised of the remaining components of index larger than 2 lies in a range in $[\frac{\pi}{2}, \frac{3\pi}{2}]$, meaning the latter is pointing in the hemisphere opposite to the direction of the major axis in such a 3-D space.

3 Conclusion

The lower bound of the modulus of the Dirichlet eta function derived in the present manuscript from 2-D principal component analysis is $\forall s \in \mathbb{C}$ s.t. $\Re(s) \in \mathcal{P}(s)$, $|\eta(s)| \geqslant |1 - \frac{\sqrt{2}}{2^\alpha}|$, where $\mathcal{P}$ is a partition spanning one half of

the critical strip, on either sides of the critical line depending upon a bivariate function represented by a letter ẓ ([z'i] IPA - Kiel) as a variable delimiting the regions complementary by mirror symmetry with respect to $\Re(s) = 1/2 \in \mathbb{C}$, and where η is the Dirichlet eta function. As a proxy of the Riemann zeta function for zero finding in the critical strip $\Re(s) \in]0, 1[$, the above as a floor function of the modulus of the Dirichlet eta function, involves the Riemann zeta function does not have zeros on either sides of the critical strip, meaning non-trivial zeros lie on the critical line $\Re(s) = 1/2$ (i.e. the Riemann hypothesis). Further observations are made, with respect to orthogonality between components of the Dirichlet eta function at non-trivial zeros of the function. Moreover, the Riemann hypothesis also known as the 8th Hilbert's problem, has strong connections with Hilbertian spaces as shown in the present work.

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